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Algebraic structure and nonclassical properties of the negative hypergeometric state

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Abstract. A new class of the *intermediate* state of the quantized radiation field, i.e. the *negative hypergeometric state* (NHGS), is introduced and its algebraic structure and nonclassical properties are investigated. The importance of NHGS lies in the fact that it can help unify such important states as the binomial, negative binomial, coherent, number and quasi-thermal states etc in quantum optics.

1. Introduction

In recent years, there has been much concern about so-called *intermediate states* in the context of quantum optics. Such a state interpolates between two other distinctive states, reducing to them in two different limits of the parameters involved. In consequence, a unifying role is played by an intermediate state in describing the physical properties of its limiting states. The earliest example in the literature is the binomial state (BS) [1], which interpolates between the coherent state and the number (Fock) state. Another is the negative binomial state (NBS) [2], which bridges the coherent state and the quasi-thermal state (i.e. Susskind–Glogower phase state [3]). The notion of BS was then generalized to the even-BS [4] (between the even-coherent and the even-number state) and the q-deformed BS [5] (between the q-coherent and the q-number state). The NBS was also generalized to the even (odd) NBS [6], which interpolates between the even (odd) coherent state and the even (odd) quasi-thermal state. The logarithmic state, as a special case of NBS with the $n = 0$ term removed, was also investigated [7]. There are also many other intermediate states, among which one can cite: (i) the generalized geometric state [8], between the number state and the (nonpure) chaotic (thermal) state; (ii) the intermediate number phase state [9], between the number state and the Pegg–Barnett phase state [10]; (iii) the intermediate number squeezed state [11], between the number state and the squeezed coherent state; (iv) the even and odd intermediate number squeezed state [12], between the even (odd) number state and even (odd) squeezed state. Most of the theoretical studies concerning these states have focused on their constructions and the possible occurrence of various nonclassical effects exhibited by them.

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Theoretically, the BS (NBS) is constructed by a linear combination of number states with coefficients chosen such that the photon-counting distribution is binomial (negative binomial). In a similar way, this paper introduces a new class of the intermediate state, to be called the *negative hypergeometric state* (NHGS), whose photon-counting distribution is the *negative hypergeometric distribution* (NHGD). (For an elementary introduction of the NHGD see the appendix. For more details, see, e.g., [13].) The NHGD is not as well known as the binomial and negative binomial distributions in classical probability theory. Nevertheless, we find some attractive features of the field whose photon-number statistics is characterized by NHGD. In fact, in two different limits of the parameters involved, the NHGS coincides with the vacuum state and the number state, respectively. More strikingly, in other two different limits, the NHGS respectively reduces to the BS and the NBS. Put in another way, the NHGS can be viewed as the *intermediate BS–NBS*. In further limiting cases, the NHGS degenerates to the coherent and the number state (from the BS) and the coherent and the quasi-thermal state (from the NBS). Therefore, the NHGS is useful in carrying out a systematic study of the properties of all these fundamental states in quantum optics.

In the next section we define the NHGS and discuss its asymptotic behaviour. In view of the fact that the BS (NBS) can be identified with a special $su(2)$ ($su(1,1)$) coherent state via the Holstein–Primakoff (HP) realization of the $su(2)$ ($su(1,1)$) generators [14, 15], we naturally expect that the algebra associated with NHGS is a deformation of both $su(2)$ and $su(1,1)$. To verify this, in section 3 we shall study the algebraic structure of NHGS, from the perspective of its ladder operator formalism. In section 4, we examine the conditions under which the NHGS exhibits nonclassical properties, including the sub-Poissonian distribution, antibunching character and squeezing effect. As expected, these properties display intermediary behaviours. The Wigner function of NHGS will also be investigated, for it helps provide insight to the nonclassical nature of the radiation field in NHGS. Some remarks on the possibility for generating the NHGS are presented in section 5. The original definition of NHGD is given in the appendix.

2. NHGS and its asymptotic properties

The NHGS is defined as a linear combination of $(M+1)$ numbers states as

$$|M,\beta,s\rangle = \sum_{n=0}^{M} \Theta_n^M(\beta,s)|n\rangle \tag{1}$$

where

$$\Theta_n^M(\beta,s) = \left[\binom{n+s}{n}\binom{M/(1-\beta)-n-s-1}{M-n}\binom{M/(1-\beta)}{M}^{-1}\right]^{1/2} \tag{2}$$

$|n\rangle$ are the number states, β is a non-negative real number less than one and s a non-negative integer satisfying

$$s < M\beta/(1-\beta). \tag{3}$$

Here the definition of the generalized binomial coefficient is

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \tag{4}$$

in which α need not be an integer. It is easy to see that in the special case $M=1$, the NHGS is actually a BS, i.e. $\sqrt{1-(1-\beta)(1+s)}|0\rangle + \sqrt{(1-\beta)(1+s)}|1\rangle$.

The name of NHGS originates from the fact that their photon-counting distribution, i.e., $|\langle n|M,\beta,s\rangle|^2 = [\Theta_n^M(\beta,s)]^2$ is the NHGD [13] in probability theory. The normalization of the NHGS can be seen from the combinatorial relation

$$\sum_{n=0}^{M} \binom{n+x}{n}\binom{M-n+y}{M-n} = \binom{M+x+y+1}{M} \tag{5}$$

namely,

$$\langle M,\beta,s|M,\beta,s\rangle = \sum_{n=0}^{M} \binom{n+s}{n}\binom{M/(1-\beta)-n-s-1}{M-n}\binom{M/(1-\beta)}{M}^{-1} = 1. \tag{6}$$

Equation (5) can be derived by making use of the formula $(1+u)^{-(z+1)} = \sum_{n=0}^{\infty}\binom{n+z}{n}(-u)^n$ and equating the power series expansion coefficients of u on the both sides of $(1+u)^{-x}(1+u)^{-y} = (1+u)^{-(x+y)}$.

The different limits, to which the three parameters M, s, and β in equation (1) go, correspond to different states in the quantized radiation field.

(a) In the limit $\beta \to 1$, M and s finite, the NHGS degenerates to the vacuum state $|0\rangle$. In fact, noticing $M/(1-\beta) \to \infty$, we have

$$[\Theta_n^M(\beta,s)]^2 \to \binom{n+s}{n}\frac{[M/(1-\beta)]^{M-n}}{(M-n)!}\frac{M!}{[M/(1-\beta)]^M} \to \delta_{n,0}. \tag{7}$$

(b) In the limit $\beta \to s/(M+s)$, $[\Theta_n^M(\beta,s)]^2 \to \binom{n+s}{n}\binom{M-n-1}{M-n}\binom{M+s}{M}^{-1} = \delta_{n,M}$, and hence the NHGS reduces to the number state $|M\rangle$.

From (a) and (b), the NHGS can be continuously transferred from the vacuum state $|0\rangle$ to the number state $|M\rangle$ by varying the value of β from $\beta = s/(M+s)$ to $\beta = 1$.

(c) In the limit $\beta \to 1$, $s \to \infty$ but keeping $(1-\beta)(1+s) = \alpha^2$ constant ($\alpha^2 < M$) (we call this the *BS limit*), we have

$$\begin{aligned}
[\Theta_n^M(\beta,s)]^2 = {} & \frac{M!}{(M-n)!}\frac{(n+s)(n+s-1)\ldots(s+1)(1-\beta)^n}{n!} \\
& \times\{[M-(1-\beta)(n+s+1)][M-(1-\beta)(n+s+2)]\ldots \\
& [M-(1-\beta)(M+s)]\} \\
& \times\{M[M-(1-\beta)][M-2(1-\beta)]\ldots[M-(M-1)(1-\beta)]\}^{-1} \\
\to {} & \binom{M}{n}\frac{(\alpha^2)^n(M-\alpha^2)^{M-n}}{M^M} = \binom{M}{n}(\alpha^2/M)^n(1-\alpha^2/M)^{M-n}
\end{aligned} \tag{8}$$

which means that the NHGS approaches the ordinary BS, i.e. $|M,\beta,s\rangle \to \sum_{n=0}^{M}[\binom{M}{n}p^n(1-p)^{M-n}]^{1/2}|n\rangle = |M,p\rangle (p \equiv \alpha^2/M$, note $\alpha^2 < M)$.

(d) In the limit $M \to \infty$, while β and s are finite (we call this the *NBS limit*),

$$\begin{aligned}
[\Theta_n^M(\beta,s)]^2 = {} & \binom{n+s}{n}\frac{M!}{(M-n)!} \\
& \times\frac{[M\beta/(1-\beta)][M\beta/(1-\beta)-1]\ldots[M\beta/(1-\beta)-s]}{[M/(1-\beta)][M/(1-\beta)-1]\ldots[M/(1-\beta)-n-s]} \\
\to {} & \binom{n+s}{n}\frac{M^n[M\beta/(1-\beta)]^{1+s}}{[M/(1-\beta)]^{1+s+n}} = \binom{n+s}{n}\beta^{1+s}(1-\beta)^n
\end{aligned} \tag{9}$$

which is none other than the negative binomial distribution. Thus in this limit the NHGS coincides with the NBS, the latter being defined as $|\beta,s\rangle = \sum_{n=0}^{\infty}[\binom{n+s}{n}\beta^{1+s}(1-\beta)^n]^{1/2}|n\rangle$.

All the facts above imply that the NHGS provides a way of treating the relations amongst different important states in quantum optics.

3. Algebraic structure of the NHGS

Recall that the NBS can be viewed as a special Perelomov's $su(1,1)$ coherent state [15, 16]

$$|\beta, s\rangle = \mathrm{e}^{r(K_+ - K_-)}|0\rangle \tag{10}$$

in which r is a positive number satisfying $\mathrm{sech}^2 r = \beta$. $K_+ = \sqrt{N+s}a^\dagger$ and $K_- = K_+^\dagger$ along with $K_3 = N + \frac{s+1}{2}$ are generators of $su(1,1)$ Lie algebra via its HP realization with the Bargmann index $(s+1)/2$. Hereafter, a and $a^\dagger$ represent the annihilation and creation operators of a photon of a single-mode electromagnetic field ($[a, a^\dagger] = 1$) and $N = a^\dagger a$ is the number operator. The Perelomov's $su(1,1)$ coherent state admits the ladder operator formalism [17], which makes us arrive at the fact that the NBS is the eigenstate of the following eigenvalue equation:

$$\left(\sqrt{1-\beta}K_+ - K_3\right)|\beta, s\rangle = -\frac{s+1}{2}|\beta, s\rangle. \tag{11}$$

In terms of the generators K_3 and K_-, the eigenvalue equation is given by

$$\left(K_- - \sqrt{1-\beta}K_3\right)|\beta, s\rangle = \frac{s+1}{2}\sqrt{1-\beta}|\beta, s\rangle. \tag{12}$$

Naturally, we ask if the NHGS can be expressed in a ladder operator formalism. The answer is affirmative. To see this, we suppose that the NHGS satisfies an eigenvalue equation as follows:

$$[f(N)a^\dagger - N]|M, \beta, s\rangle = 0 \tag{13}$$

in which $f(N)$ is a function of N to be determined. Inserting equation (1) into equation (13) and taking into account the well known relations for the action of boson operators onto the Fock states (e.g. [18]), we obtain the following equation:

$$\sum_{n=0}^{M} f(n+1)\sqrt{n+1}\Theta_n^M(\beta, s)|n+1\rangle - \sum_{n=0}^{M} n\Theta_n^M(\beta, s)|n\rangle = 0 \tag{14}$$

from which we find

$$\begin{aligned} &f(M+1) = 0 \\ &f(n) = \frac{\sqrt{n}\Theta_n^M(\beta, s)}{\Theta_{n-1}^M(\beta, s)} = \frac{\sqrt{1-\beta}\sqrt{n+s}\sqrt{M-n+1}}{\sqrt{M-(1-\beta)(n+s)}} \qquad (n = 1, 2, \ldots, M). \end{aligned} \tag{15}$$

Substituting equation (15) into equation (13) gives the result

$$\left[\frac{\sqrt{1-\beta}\sqrt{N+s}\sqrt{M-N+1}}{\sqrt{M-(1-\beta)(N+s)}}a^\dagger - N\right]|M, \beta, s\rangle = 0. \tag{16}$$

This equation obviously reduces to equation (11) in the NBS limit. In the BS limit, on the other hand, equation (16) reduces to the following equation which is satisfied by the BS $|M, p\rangle$ $(p = \alpha^2/M)$:

$$\left(\sqrt{p}J_- + \sqrt{1-p}J_z\right)|M, p\rangle = \tfrac{1}{2}\sqrt{1-p}M|M, p\rangle \tag{17}$$

in which $J_- = a^\dagger\sqrt{M-N}$, $J_+ = J_-^\dagger$ and $J_z = \frac{M}{2} - N$ are the HP realization of $su(2)$ Lie algebra.

Consider now the operators

$$N, A_+ = \frac{\sqrt{1-\beta}\sqrt{N+s}\sqrt{M-N+1}}{\sqrt{M-(1-\beta)(N+s)}}a^\dagger \qquad A_- = (A_+)^\dagger. \tag{18}$$

They satisfy the following commutation relations:

$$[N, A_\pm] = \pm A_\pm \qquad A_+A_- = F(N) \qquad A_-A_+ = F(N+1) \tag{19}$$

where

$$F(N) = \frac{(1-\beta)(N+s)(M-N+1)N}{M-(1-\beta)(N+s)} \tag{20}$$

is a non-negative Hermitian function. Therefore, the algebra related to NHGS, which is an associative algebra generated by A_-, A_+, N and the unit 1, is actually a *generally deformed oscillator algebra* with the structure function $F(N)$ [19]. Since $A_+|M\rangle = 0$, the algebra has an $(M+1)$-dimensional representation on the Fock space. In terms of the generators of this algebra, equation (16) is rewritten as

$$[A_+ - N]|M, \beta, s\rangle = 0. \tag{21}$$

Following the same procedure as in deriving equation (16), we find another form of the eigenvalue equation satisfied by the NHGS:

$$\left[a\sqrt{N+s} - \frac{\sqrt{1-\beta}\sqrt{M-N}(N+s+1)}{\sqrt{M-(1-\beta)(N+s+1)}}\right]|M, \beta, s\rangle = 0 \tag{22}$$

which in the NBS limit tends to equation (12). In the BS limit it tends to the eigenvalue equation in terms of J_z and J_+:

$$\left(\sqrt{p}J_z - \sqrt{1-p}J_+\right)|M, p\rangle = -\tfrac{1}{2}\sqrt{p}M|M, p\rangle. \tag{23}$$

From equations (17) and (23), the BS can be identified with a special $su(2)$ coherent state, as has been indicated in [14]. In a word, the algebra associated with the NHGS is a deformation of both $su(2)$ and $su(1, 1)$.

4. Nonclassical properties

4.1. Photon statistics

A field is antibunched if its second-order correlation function $g^{(2)}(0) < 1$ [20], namely,

$$g^{(2)}(0) = \frac{\langle N^2\rangle - \langle N\rangle}{\langle N\rangle^2} < 1. \tag{24}$$

For the NHGS, the averages $\langle N\rangle$, $\langle N^2\rangle$ and the fluctuation $\langle \Delta N^2\rangle$ are given by the following expressions:

$$\begin{aligned}
\langle N\rangle &= \frac{(s+1)(1-\beta)M}{1-\beta+\beta M}\\
\langle N^2\rangle &= \frac{(s+1)(1-\beta)M}{1-\beta+\beta M} + \frac{(s+1)(s+2)(1-\beta)^2M(M-1)}{(1-\beta+\beta M)(2-2\beta+\beta M)}\\
\langle \Delta N^2\rangle &= \frac{(s+1)(1-\beta)M}{1-\beta+\beta M} + \frac{(s+1)(s+2)(1-\beta)^2M(M-1)}{(1-\beta+\beta M)(2-2\beta+\beta M)} - \frac{(s+1)^2(1-\beta)^2M^2}{(1-\beta+\beta M)^2}
\end{aligned} \tag{25}$$

from which we find

$$g^{(2)}(0) = \frac{(s+2)(1-\beta+\beta M)(M-1)}{(s+1)(2-2\beta+\beta M)M}. \tag{26}$$

It then follows for $g^{(2)}(0) < 1$, $s > \frac{\beta(M-1)^2+\beta-2}{1-\beta+M}$. Combining this condition with equation (3), we arrive at the sufficient and necessary condition for the NHGS to be antibunched (for given β and M)

$$\frac{\beta(M-1)^2+\beta-2}{1-\beta+M} < s < \frac{M\beta}{1-\beta}. \tag{27}$$

For fixed β and s, it is evident that the left-hand side of the above inequality will be destroyed when M becomes large enough. The extreme situation occurs in the NBS limit: $g^{(2)}(0) \to \frac{s+2}{s+1} > 1$. In the BS limit, on the other hand, we have $g^{(2)}(0) \to 1 - \frac{1}{M} < 1$. Therefore the NBS is definitely bunched while the BS is definitely antibunched.

Since the sub-Poissonian character and antibunching effect are always coincident for single-mode and time-independent fields, equation (27) is also the condition for the NHGS to be sub-Poissonian.

Now we discuss how $g^{(2)}(0)$ changes as we vary the parameters in the NHGS. In the special case $M = 1$, from equation (26) we see $g^{(2)}(0) = 0$, which is an expected result since the state $|1, \beta, s\rangle$ does not contain in its expansion the photon number state $|2\rangle$. In another special case $M \to \infty$ and $s = 0$, we find that $g^{(2)}(0) \to 2$, in agreement with the value obtained in the quasi-thermal state, as it should. In our numerical study of $g^{(2)}(0)$, we plot this function against the parameter β for different values of M and s (see figure 1). (Here and henceforth, the condition in equation (3) is made use of to decide the starting points of β.) From these plots we find that when $\beta \to s/(M+s)$, $g^{(2)}(0) \to (1 - 1/M)$ (the number state); when $\beta \to 1$, $g^{(2)}(0) \to \frac{(1-1/M)(s+2)}{(s+1)}$. It may be recalled that the latter case corresponds to the vacuum state. Thus we conclude that the second-order correlation function would be undefined for the vacuum state. Other conditions being the same, $g^{(2)}(0)$ is a monotonically increasing function of β. Accordingly, the maximum antibunching is achieved in the number state. Also, we find that $g^{(2)}(0)$ decreases with increasing the value of s. When $s < M - 2$, there is an interval of β within which the NHGS exhibits the bunching effect. When $s \geqslant M-2$, the antibunching behaviour persists for the whole interval of β. Finally, $g^{(2)}(0)$ increases as M increases. This means that increasing the number of photons in the NHGS may change the field from antibunching to bunching.

4.2. Quadrature squeezing

The quadrature operators of the single-mode field are defined as

$$X = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad P = \frac{1}{\sqrt{2}\mathrm{i}}(a - a^\dagger). \tag{28}$$

They satisfy the commutation relation $[X, P] = \mathrm{i}$ and consequently their variances $(\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2$, $(\Delta P)^2 = \langle P^2\rangle - \langle P\rangle^2$ obey the Heisenberg uncertainty relation

$$(\Delta X)^2(\Delta P)^2 \geqslant \tfrac{1}{4}. \tag{29}$$

The field is said to be squeezed in the X (P) quadrature if $(\Delta X)^2 < \frac{1}{2}$ ($(\Delta P)^2 < \frac{1}{2}$). Seeing that there is no squeezing in the X quadrature when the field is in the NBS [15], whereas the BS can exhibit the X-squeezing effect when appropriate values of its parameters are taken [1], the NHGS should be able to exhibit X-squeezing in some regions of its parameters; such regions are not close to the NBS limit. For the sake of convenience, we define the squeezing indices as

$$S_x = 2(\Delta X)^2 - 1 \qquad S_p = 2(\Delta P)^2 - 1. \tag{30}$$

When $S_x < 0$ ($S_p < 0$), the field is squeezed in the X (P) quadrature. From equation (28), S_x and S_p are expressed as

$$S_x = \langle a^2 + a^{\dagger 2}\rangle + 2\langle N\rangle - \langle a + a^\dagger\rangle^2 \tag{31}$$

$$S_p = -\langle a^2 + a^{\dagger 2}\rangle + 2\langle N\rangle + \langle a - a^\dagger\rangle^2. \tag{32}$$

Now that all parameters in the definition of the NHGS are real, we have $\langle a\rangle = \langle a^\dagger\rangle$ and $\langle a^2\rangle = \langle a^{\dagger 2}\rangle$. Hence,

$$S_x = 2\langle a^2\rangle + 2\langle N\rangle - 4\langle a\rangle^2 \qquad S_p = -2\langle a^2\rangle + 2\langle N\rangle. \tag{33}$$

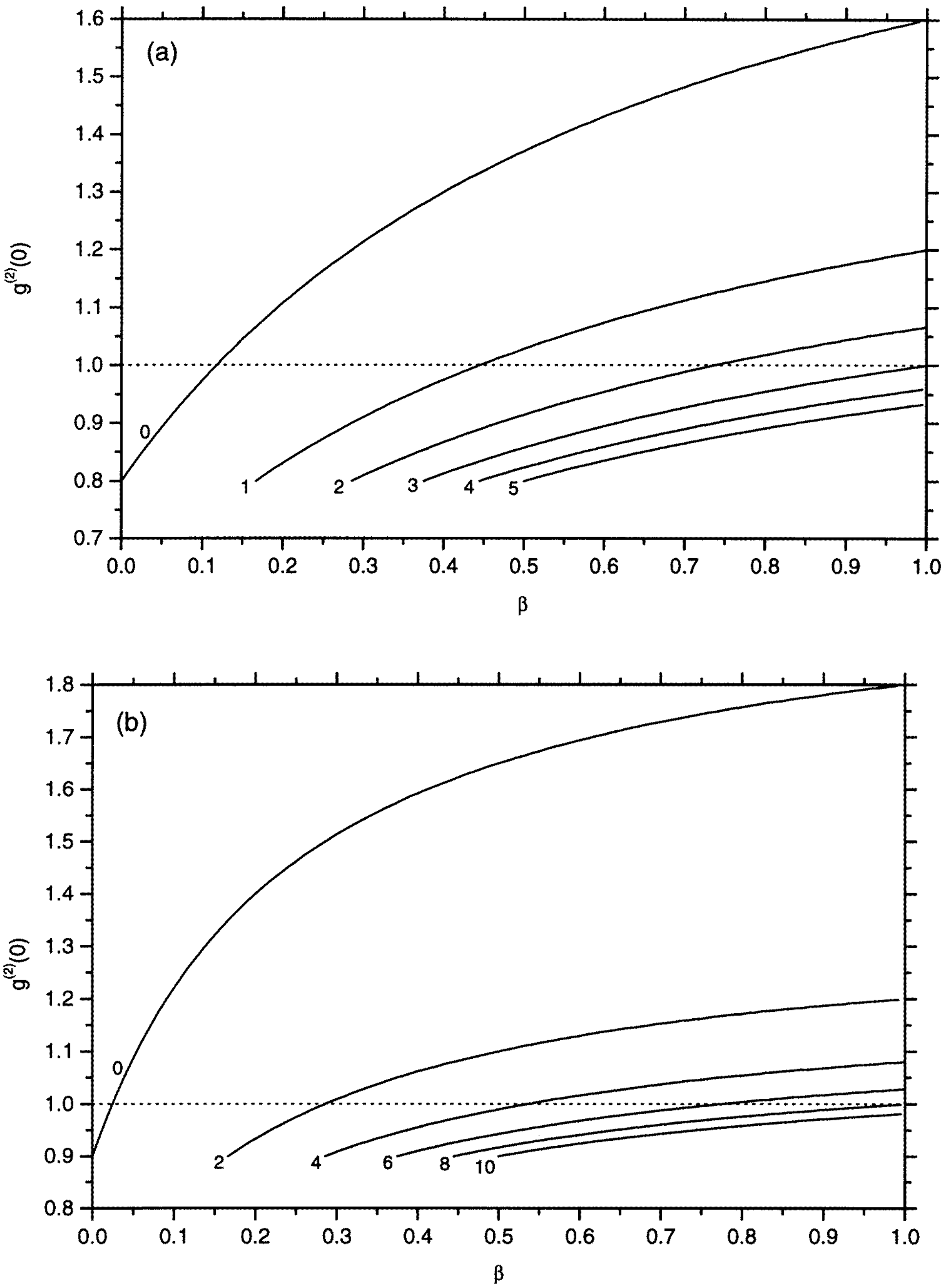

Figure 1. The second-order correlation function of the NHGS as a function of β for two different values of M: (*a*) $M = 5$ and (*b*) $M = 10$. The s values are indicated in the plots.

The action of powers of the annihilation operator on the NHGS is

$$a^l|M,\beta,s\rangle = \sum_{n=0}^{M-l} \Theta_{n+l}^{M}(\beta,s)\sqrt{n+l}\sqrt{n+l-1}\dots\sqrt{n+1}|n\rangle \tag{34}$$

when $l \leqslant M$. When $l > M$, $a^l|M, \beta, s\rangle = 0$. First, we consider the case $M = 1$. In this case it is easy to evaluate

$$S_x = -2(1+s)(1-\beta) + 4(1+s)^2(1-\beta)^2 \qquad S_p = 2(1+s)(1-\beta) \tag{35}$$

from which we see that the quadrature P is not squeezed; the occurrence of squeezing in the quadrature X demands

$$s < \frac{2\beta - 1}{2(1-\beta)}. \tag{36}$$

Second, when $M > 1$, by means of equations (33) and (34) we get the expressions of the squeezing indices as

$$S_x = 2\frac{(s+1)(1-\beta)M}{1-\beta+\beta M} + 2\sum_{n=0}^{M-2} \Theta_{n+2}^M(\beta, s)\Theta_n^M(\beta, s)\sqrt{n+2}\sqrt{n+1} - 4\left(\sum_{n=0}^{M-1} \Theta_{n+1}^M(\beta, s)\Theta_n^M(\beta, s)\sqrt{n+1}\right)^2 \tag{37}$$

$$S_p = 2\frac{(s+1)(1-\beta)M}{1-\beta+\beta M} - 2\sum_{n=0}^{M-2} \Theta_{n+2}^M(\beta, s)\Theta_n^M(\beta, s)\sqrt{n+2}\sqrt{n+1}. \tag{38}$$

It is convenient to make numerical evaluations of the expressions above. We have plotted S_x in figure 2, against the parameter β for different values of M and s. One can see that the NHGS does exhibit squeezing in the quadrature X when the value of s is not too small ($s \geqslant 2$ in the case $M = 5$, for instance). However, the depth of squeezing and the range of β over which squeezing is observed are very sensitive to the values of M and s: as s increases, the X-squeezing becomes more effective, and the point of maximum squeezing shifts to higher β. As a result, the most effective X-squeezing is achieved in the BS limit. In figure 4 we have plotted S_x, by letting $M = 5$ and $s = 999$, against $(1+s)(1-\beta)/M$, which approximates to S_x for the BS of $M = 5$. Comparing figures 4 and 2(a), it is evident that the maximum X-squeezing of the BS is stronger than that of the NHGS. For the quadrature P, however, the opposite is the case. For example, only when the value of s is taken small enough ($s \leqslant 2$ in the case $M = 5$, for instance) does P-squeezing appear (see figure 3), and P-squeezing becomes more effective as s decreases, namely, maximum P-squeezing is achieved when $s = 0$. On the other hand, for given values of β and s, as M increases, S_x increases while S_p decreases. This means that increasing the photon number can enhance the squeezing in the P quadrature but depress that in the X quadrature. At this point we may naturally conclude that in the NBS limit there is no squeezing at all in the X quadrature, whereas the P quadrature is always squeezed irrespective of the values of s and β. This is consistent with the results in [15].

In figure 5 we have plotted the uncertainty product $(\Delta X)^2(\Delta P)^2$ for $M = 5$, against β for different s values. The uncertainty product increases as s increases. It is always greater than its minimum allowed value $\frac{1}{4}$. This value is achieved only in the limit $\beta \to 1$, i.e. the limit leading to the vacuum state.

4.3. Wigner function

Quasi-probability distributions can help provide insight to the nonclassical nature of radiation fields. Of these, the Wigner function [21] is of particular importance as it gives the correct probability for a chosen observable by integration over the conjugate observable. The Wigner function is defined as the Fourier transform of the characteristic function, associated with

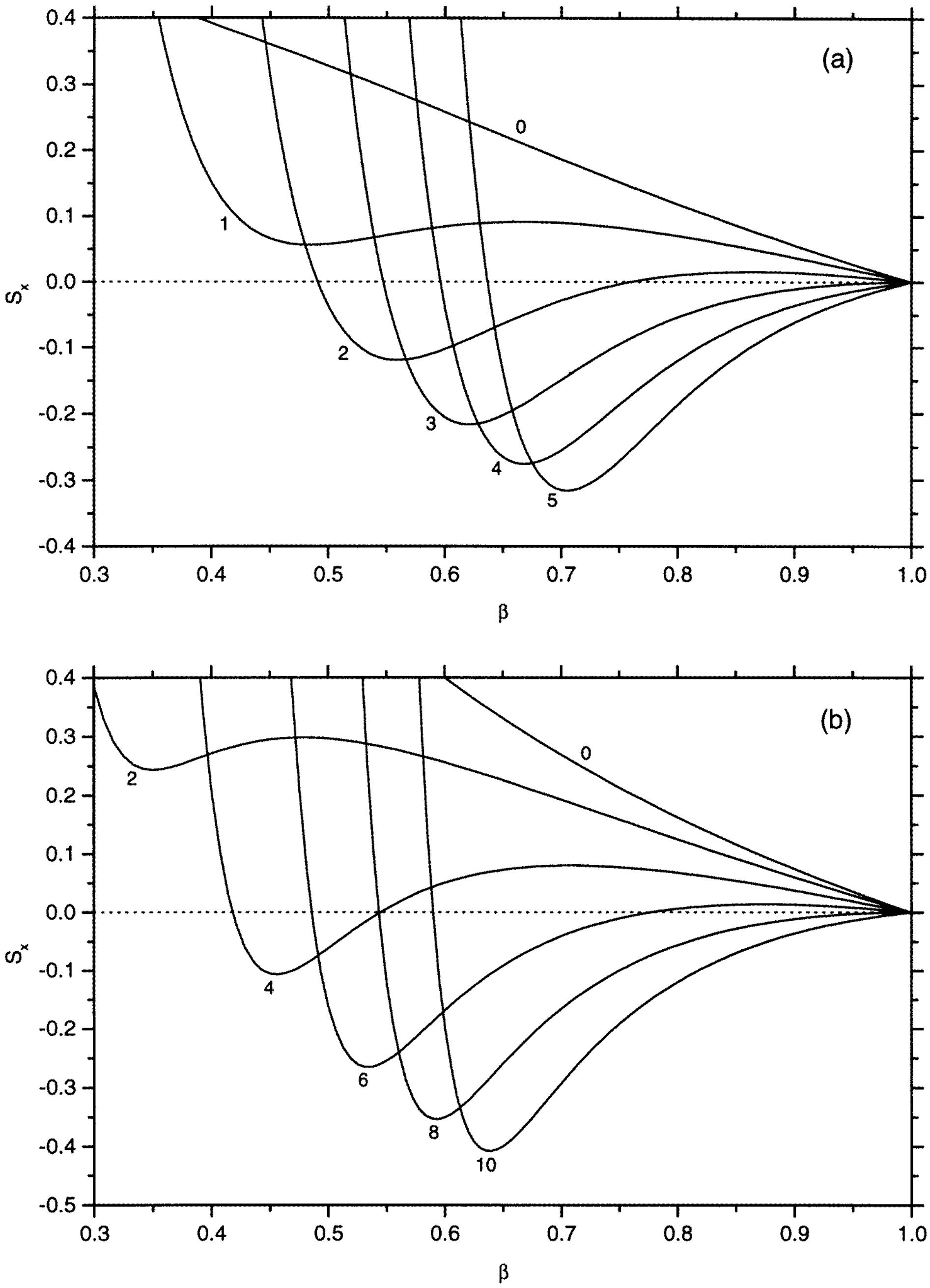

Figure 2. Squeezing index S_x of NHGS as a function of β for two different values of M: (a) $M = 5$ and (b) $M = 10$. The s values are indicated in the plots.

the symmetrical order of the annihilation and creation operators. Alternatively, the Wigner function for an arbitrary density operator ρ may be given by [22]

$$W(z) = \frac{2}{\pi}\,\mathrm{Tr}[\rho D(2z)\exp(\mathrm{i}\pi N)] \tag{39}$$

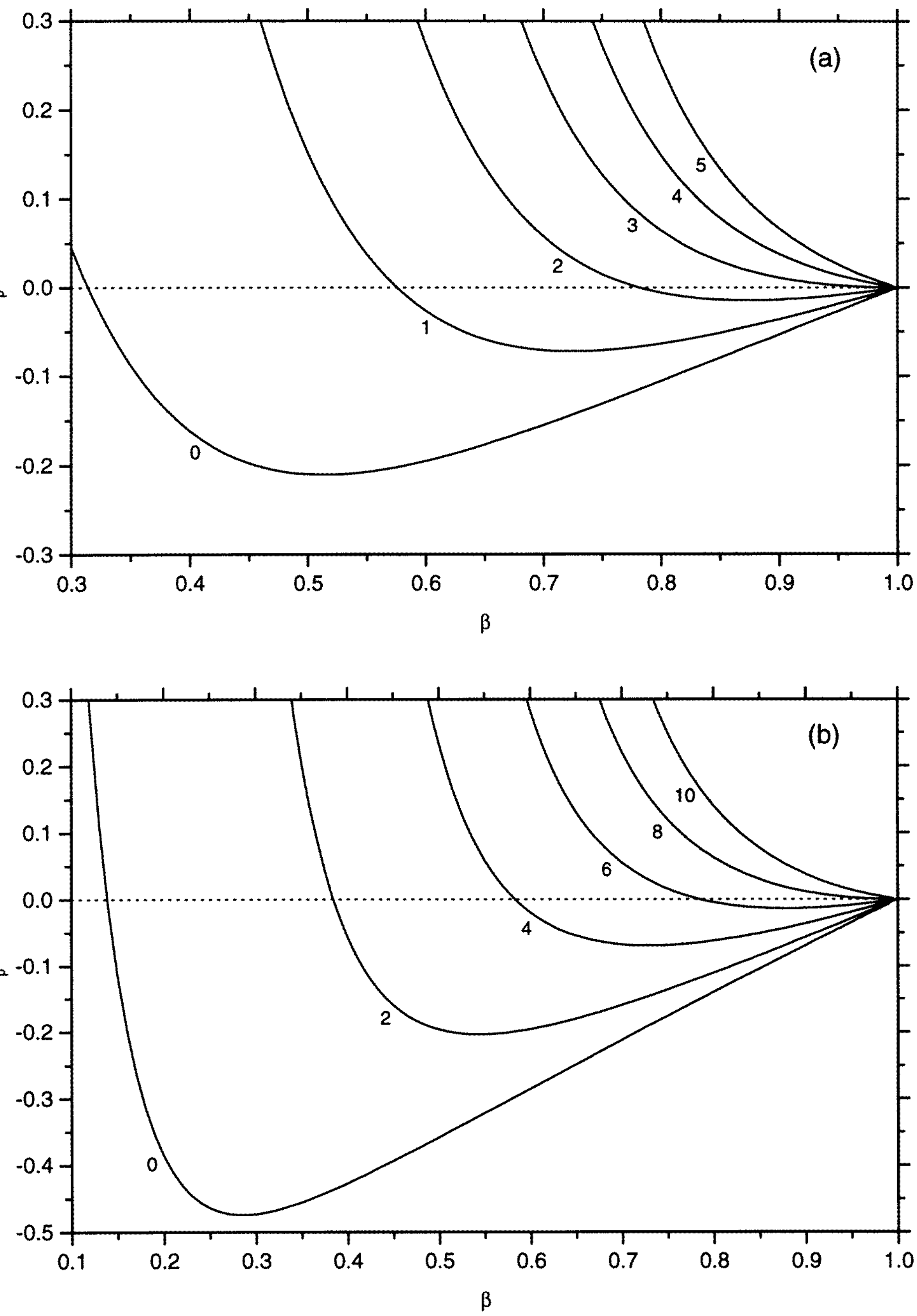

Figure 3. Squeezing index S_p of NHGS as a function of β for two different values of M: (a) $M = 5$ and (b) $M = 10$. The s values are indicated in the plots.

where $D(z)$ is the displacement operator, z is a complex c-number. Inserting $\rho = |M, \beta, s\rangle\langle M, \beta, s|$ and substituting the definition of equation (1), we obtain the Wigner function

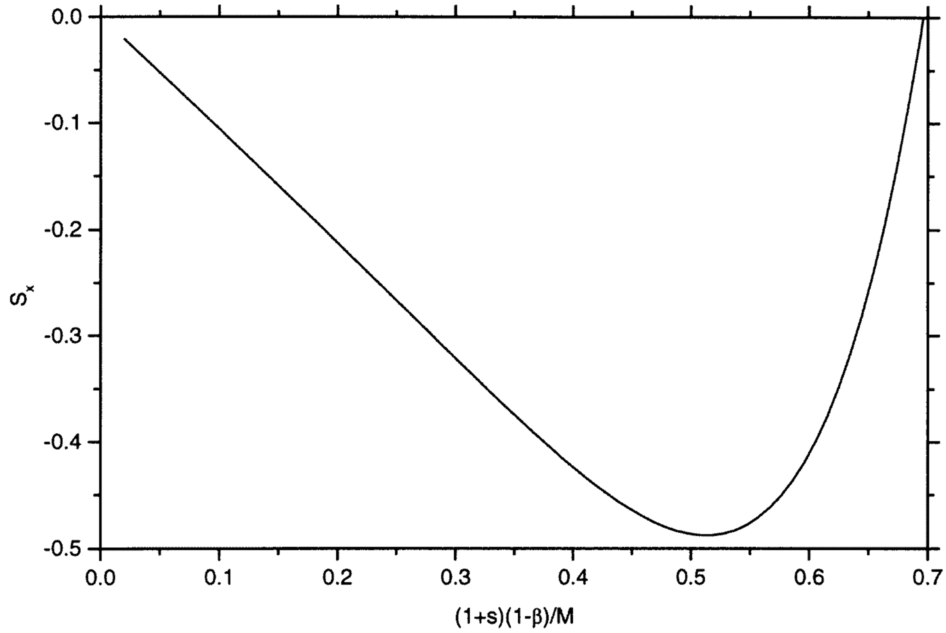

Figure 4. Squeezing index S_x of NHGS as a function of $(1+s)(1-\beta)/M$ for $M = 5$ and $s = 999$.

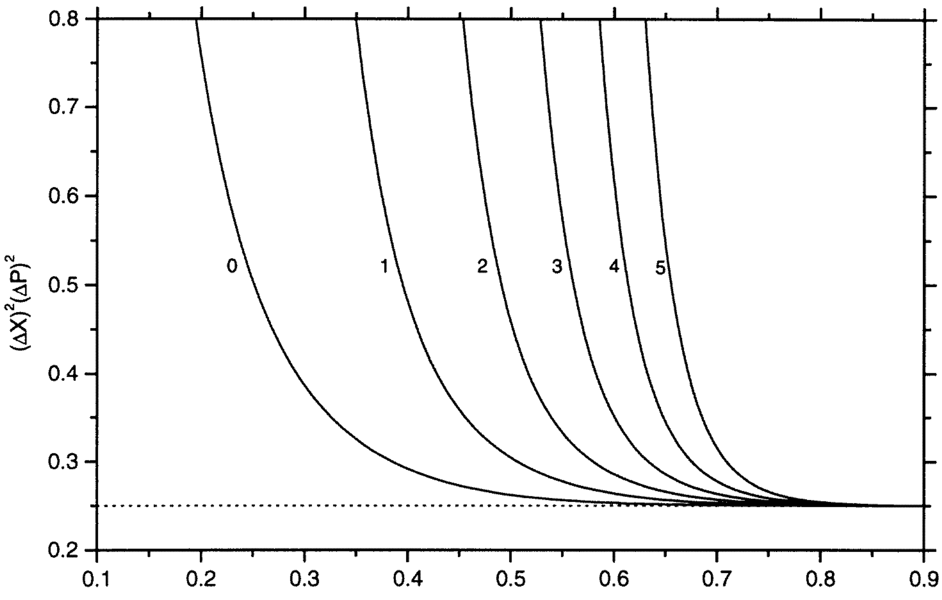

Figure 5. The uncertainty product $(\Delta X)^2(\Delta P)^2$ of NHGS as a function of β for $M = 5$. The s values are indicated in the plots.

of the NHGS as

$$W(z) = \frac{2}{\pi} \sum_{n=0}^{M} \sum_{m=0}^{M} \Theta_n^M(\beta, s)\Theta_m^M(\beta, s)(-1)^m \chi_{nm}(2z). \tag{40}$$

Here [23],

$$\chi_{nm}(z) = \langle n|D(z)|m\rangle = \langle m|D(-z)|n\rangle^* = \sqrt{\frac{m!}{n!}} \mathrm{e}^{-|z|^2/2} z^{n-m} L_m^{n-m}(|z|^2) \tag{41}$$

for $n \geqslant m$, where $L_m^\nu(x) \equiv \sum_{n=0}^{m} \binom{m+\nu}{m-l} \frac{(-x)^l}{l!}$ is the associated Laguerre polynomial. From equation (40) it is apparent that $W(z)$ is a symmetric function in $\mathrm{Im}\,(z)$. We have studied numerically the behaviour of the Wigner function $W(z)$ of equation (40) as a function of $z = \mathrm{Re}\,(z) + \mathrm{i}\mathrm{Im}\,(z)$ for $M = 2$, $s = 0$ and different values of β (from 1 to $s/(M + s)$). The result is shown in the sequence of figures 6(*a*)–(*d*). With reference to figure 6(*a*), we see that when β is close to 1, the function has an almost Gaussian shape centred in the origin, and is largely insensitive to change in M. This is reasonable because of the dominance of the effect of the vacuum state over the effects of the higher excitations. As β decreases ($\beta = 0.5$, for instance), which means that the vacuum state begins to lose its higher probability in the number state expansion, some negative part of the distribution appears. The negativity of the Wigner function signifies nonclassical effects. When the value of β is even smaller, the negative part is even larger, $W(z)$ deviates far away from the Gaussian distribution and the rings characteristic of a number state start being formed, as we can appreciate in the plot of $\beta = 0.1$. When $\beta = 0$, the number state is produced.

In figure 7 we plot the Wigner function $W(z)$ by taking the values $M = 1$, $s = 1$, $\beta = 0.55$, which is identified with the Wigner function of the BS $|M = 1, p = 0.9\rangle$. On the other hand, the numerical evaluation of $M = 1$, $s = 999$, $\beta = 0.9991$ produces a plot very much resembling that corresponding to the BS $|M = 1, p = 0.9\rangle$, which enables us to trace the 'shaping' of the BS from the NHGS. Similarly, one can also trace the 'shaping' of the NBS by plotting the Wigner function of NHGS, taking a very large value of M.

5. Concluding remarks

In this work we have introduced the NHGS of the quantized radiation field and discussed some of its physical properties. We have shown that the NHGS can be viewed as interpolating between the BS and the NBS.

Finally, let us briefly discuss the possibility for realization of the NHGS. As a matter of fact, although the generation of pure nonclassical states has been a major subject in quantum optics, it is not a task of immediate implementation. To the author's knowledge, even the realization of the BS is not available yet. Recently, some progress has been made in schemes of realizing arbitrary pure states. In [24], for example, a method based upon a *nonunitary* 'collapse' of the state vector of the cavity-field mode via atom ground-state measurement is proposed for preparing a cavity-field mode undergoing a Jaynes–Cummings dynamics in any superposition of a finite number of Fock states in principle. The scheme in [25], however, is by using a cavity QED *unitary* time-dependent interaction. With respect to these two methods, it has been argued by the authors of [26] that 'both approaches involve individual atoms interacting with a single-mode cavity field, which would demand extraordinary control in a generation experiment. It is therefore interesting to seek alternative methods for the generation of nonclassical light'. [26] constructs a Hamiltonian which would allow the use of some kind of nonlinear interaction for the production of arbitrary pure states, including

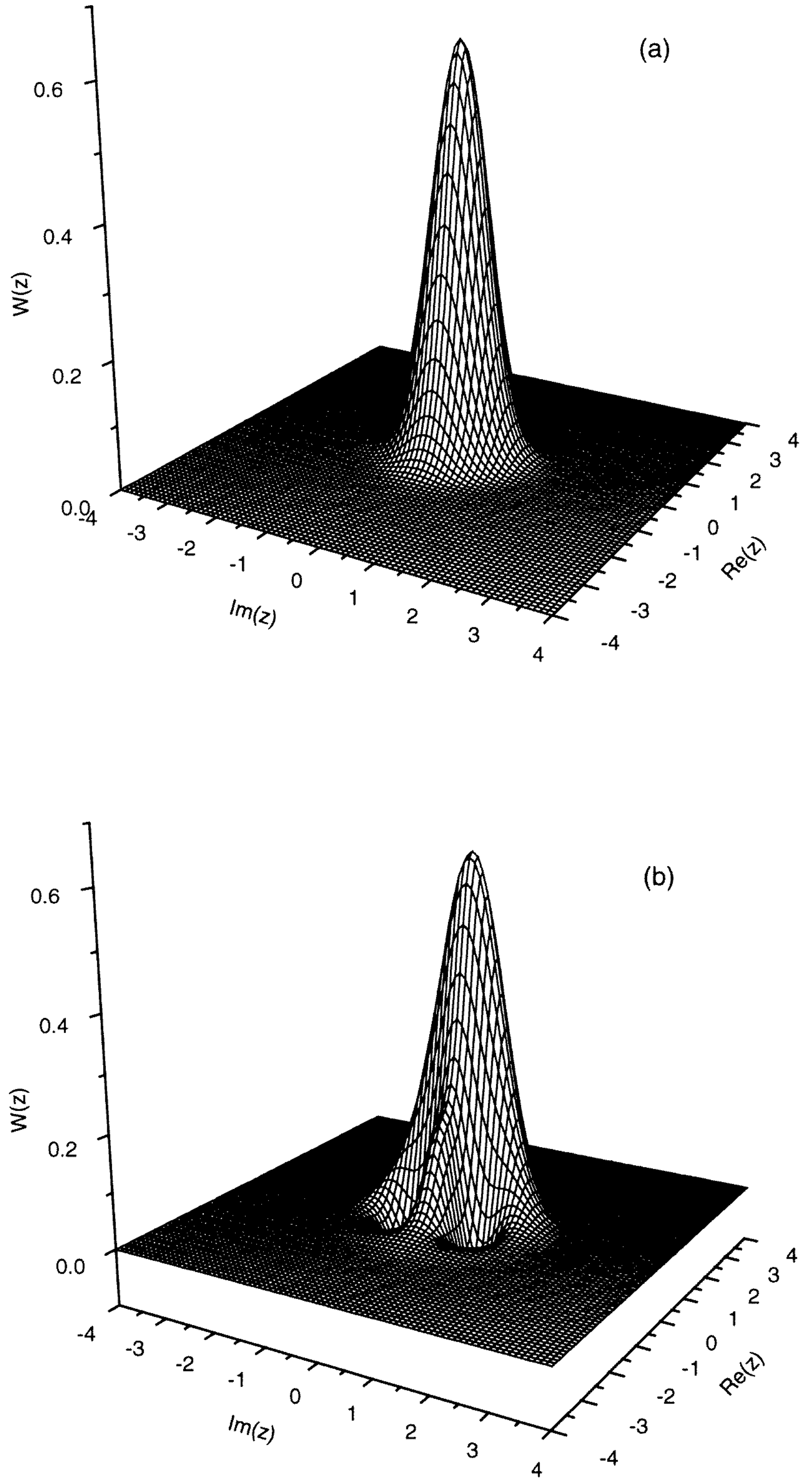

Figure 6. Wigner function $W(z)(z = \mathrm{Re}\,(z) + \mathrm{i}\mathrm{Im}\,(z))$ of the NHGS for $M = 2$, $s = 0$ and (*a*) $\beta = 0.9$, (*b*) $\beta = 0.5$, (*c*) $\beta = 0.1$, (*d*) $\beta = 0$ (the number state $|2\rangle$).

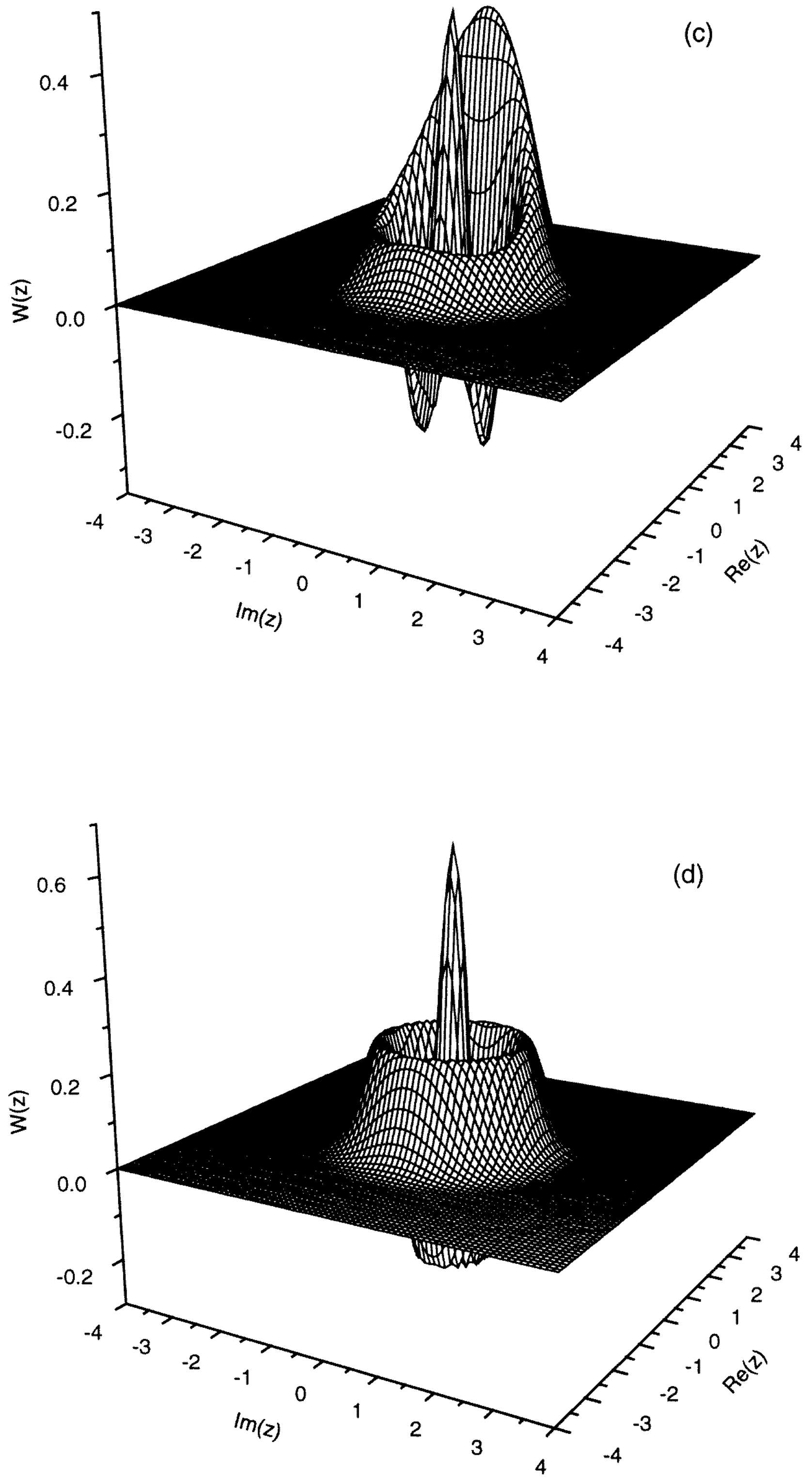

Figure 6. (Continued)

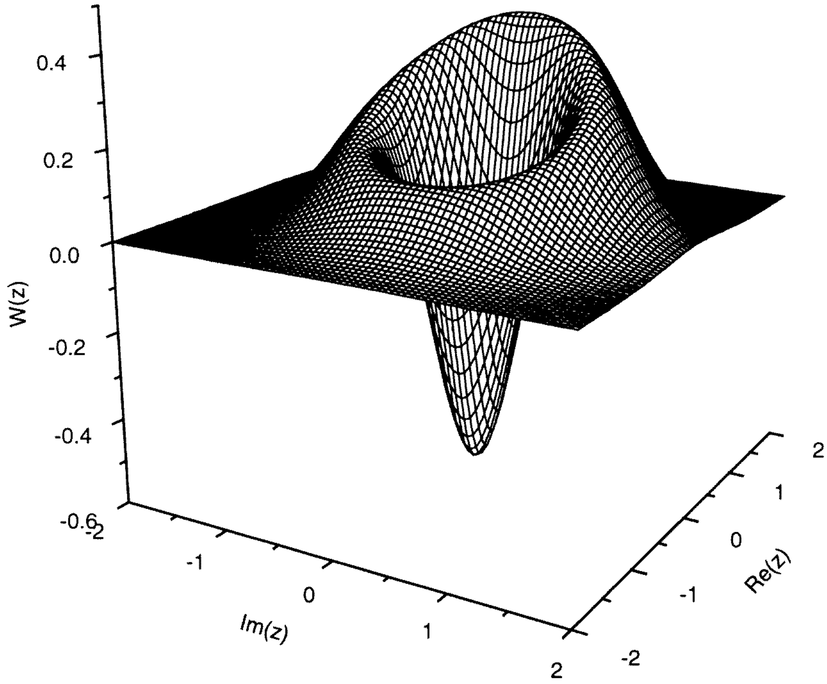

Figure 7. Wigner function $W(z)$ of the NHGS for $M = 1$, $s = 1$, $\beta = 0.55$.

the superposition states like the NHGS. In a more recent paper, i.e. [27], it is shown that arbitrary pure quantum states can be realized by a succession of alternate state displacement and single-photon adding. Based on the above significant studies, the NHGS could hopefully be produced in the not too distant future. If so, the interesting features stated in our paper will be witnessed.

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Appendix. NHGD

Assume that there are N balls, K of which are black and $(N - K)$ are white. Suppose we choose balls at random, one by one and without replacement. Let τ_r be the number of our choices needed in order to exactly obtain r black balls. Then the probability of τ_r being m $(m = r, r+1, \ldots, N-K+r)$ is given by the NHGD

$$P\{\tau_r = m\} = \binom{m-1}{r-1}\binom{N-m}{K-r}\binom{N}{K}^{-1}.$$

By a change of variables

$$n = m - r \qquad s = r - 1 \qquad M = N - K = (1-\beta)N$$

we can rewrite the probability as

$$P\{\tau_r = n + s + 1\} = \binom{n+s}{n}\binom{M/(1-\beta) - n - s - 1}{M - n}\binom{M/(1-\beta)}{M}^{-1}$$

which is just $[\Theta_n^M(\beta, s)]^2$.

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